The definitions for the time derivatives of k_2 and h_2 can be found in an earlier section. The functional behavior of Δi_{yr} is exactly that of the control strategy. Consequently, the technique also has determined the minimum annual velocity change required. As velocity change (delta-V) is directly related to propellant consumption, the minimum amount of consumable has also been found.

Conclusion

A procedure or control strategy has been formulated that produces a trajectory history permitting the maximum time between maneuvers given a constraining limit. Consequently, this also leads to the optimum (i.e., minimum) annual delta-V requirements.

There is considerable payoff from this procedure. The technique developed follows the slowly varying precession toward Aries, whereas other techniques do not. Use of these techniques has a price in the form of goal vectors in error with respect to the optimum. This error will average about 4.5 deg corresponding to increased velocity needs as much as 7% greater. For a fixed propellant load, the technique described in the control strategy section will extend mission life by this same amount (i.e., up to 6 months additional life for a 7-yr vehicle), in contrast to other strategies.

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Roll Resonance Probability for Ballistic Missiles with Random Configurational Asymmetry

M.D. Bennett*
Sandia National Laboratories, Albuquerque, New Mexico

Introduction

ROLL resonance during ballistic flight normally is a routine, uneventful state in which the missile spin rate briefly becomes resonant with the pitch rate. However, under certain conditions these two time-varying frequencies will converge and tend to remain stable with respect to each other for an extended period. When resonance persists, the aerodynamic trim angle is amplified. In response, the angle of attack rapidly increases, particularly in the case of small bodies, and as a result of the increased drag, trajectory dispersion suffers. Sometimes the lateral force developed on the vehicle becomes excessive and causes catastrophic failure. Consequently, the roll resonance event is a major design consideration of uncontrolled flight.[1]

Seemingly symmetrical missiles, including finless bodies of revolution such as conventional artillery shells and re-entry vehicles, produce small but not always negligible roll torques. The configurational irregularities, which can be brought about either by in-flight distortions caused by aerodynamic heating or loading or by manufacturing tolerances and imperfections, induce a body-fixed side force that combines with an offset or misaligned mass axis to create a moment about the vehicle roll axis. If the moment acts in a favorable direction, the resulting angular acceleration serves to further separate the roll and pitch frequencies, and an ordinary transient resonance situation takes place. In the other direction, a rolling moment can sustain the resonance mode when the torque exceeds a critical amount (L_{crt}).

In a group of supposedly identical missiles, the aerodynamic and mass asymmetries will vary from body to body in some random fashion. The sizes of the rolling moments induced by the slight configurational disorders depend on the magnitudes and relative orientations of the two factors that comprise each of the moments: the lateral force due to a nonzero aerodynamic trim angle, and a lever arm arising from nonuniform mass distribution. With a fixed degree of maximum possible asymmetry and specified flight conditions, the moments will range over the interval $\pm L$, where L denotes the finite limiting value. If the possible asymmetry becomes excessive, i.e., capable of prolonging a coupling of the roll and pitch frequencies ($|L| > |L_{crt}|$), some, but not all, of the bodies will possess a moment of sufficient size and direction to maintain resonance. Hence, a problematical or uncertain condition of resonance prevails.

To quantify a missile's susceptibility to the resonance state, an integral equation was written that describes the likelihood of the occurrence of continuous roll resonance, assuming that the asymmetry moment evolves randomly without bias. The derivation of the probability equation is presented along with numerical solutions in the following discussion. No other record of resonance probability theory is known.

Discussion

The angular acceleration in the roll direction will not be sufficient to sustain a permanent state of resonance when the torque resulting from configurational abnormalities is dominant and remains less than a critical amount.† Therefore, the probability of continuous resonance is zero for missiles with negligible asymmetry; i.e., $P=0$ if $|L| < |L_{crt}|$.

When the asymmetry moment becomes excessive, resonance may occur. Assuming that uniform flight conditions are maintained and that the angle of attack has stabilized at the trim condition, the boundary that defines the circumferential region in which a lateral center-of-mass position vector $\boldsymbol{\delta}$ will be situated during the inception of resonance is readily determined from the problem geometry. If the contribution to the trim angle that arises from mass asymmetry effects is ignored, the angle $\zeta = \cos^{-1} |\delta_{crt}| / |\delta|$ demarcates the resonance zone in the case of out-of-plane asymmetries (see Fig. 1).

The determination of the probability of resonance now becomes a matter of asymmetry apportionment. An elementary case involves a group of nearly identical missiles

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*Member of Technical Staff, Aerodynamics Department. Associate Fellow AIAA.

†Numerical studies can be conducted with arbitrary flight conditions using a multidegree-of-freedom trajectory computer code to bracket the critical moment by a trial-and-error process. Approximate, closed-form expressions of roll resonance criteria, from which the critical value may be estimated, are developed for out-of-plane asymmetries by Vaughn[2] and for in-plane asymmetries by Barbera.[3] *Out-of-plane* asymmetries refer to an arrangement in which the planes of the trim angle and the cm position vector are mutually perpendicular. These two planes contain the body longitudinal axis and are coincident with in-plane asymmetries.

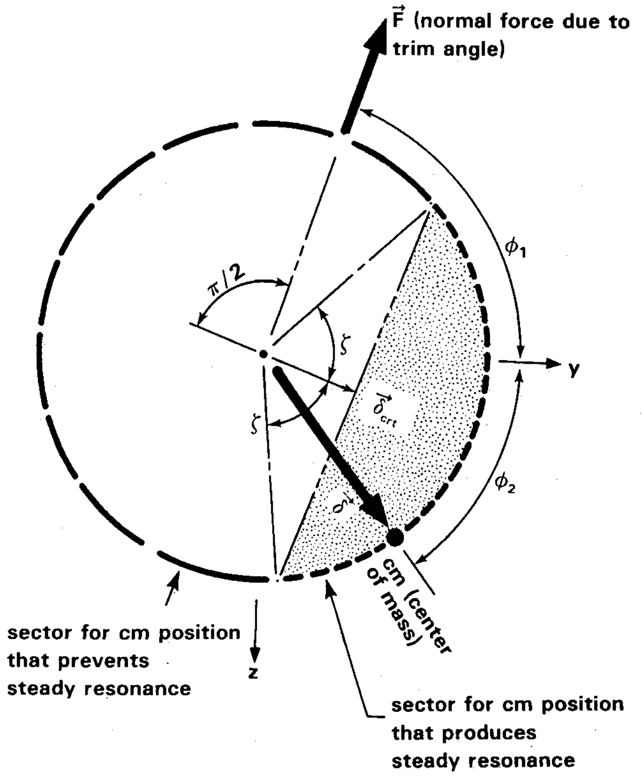

Fig. 1 View along body longitudinal axis showing the circumferential zone in which a lateral displacement δ of the missile's center of mass lies during continuous roll resonance. The product $|F\times\delta_{crt}|$ is the minimum rolling moment that will perpetuate resonance; the maximum possible moment due to the out-of-plane asymmetries is $|F|\,|\delta|$.

that have trim forces F and cm position vectors of constant magnitudes, with their relative angular positions $\phi=\phi_1+\phi_2$ uniformly distributed around the bodies. Then, the resonance probability P is determined by the ratio of the arc length of the sector that results in resonance to the circumference of the complete circle. Hence, when ϕ_1 is constant and ϕ_2 uniformly varies over the range $0\leq\phi_2\leq 2\pi$ rad, $\cos(\pi P)=|\delta_{crt}|/|\delta|$. By considering an inverse order in which the cm vector is taken to be stationary while the angular orientation of the aerodynamic trim angle uniformaly varies, a companion equation can be written in the manner $\cos(\pi P)=|\alpha_{crt}|/|\alpha|$, where $\alpha\sim|F|$ denotes the nonrolling trim angle. The general form of the constant-asymmetry probability formula, deduced from these two equations, expresses the likelihood of steady resonance in terms of a dimensionless rolling moment $\mathcal{L}=L/L_{crt}$: if $\mathcal{L}^2\geq 1$, $\cos(\pi P)=1/|\mathcal{L}|$, or,

$$P=(1/\pi)\sec^{-1}|\mathcal{L}| \tag{1}$$

As expected, Eq. (1) indicates that in the limit, where the possible moment grows very large, steady resonance becomes as likely to occur as not ($P\to\frac{1}{2}$ when $\mathcal{L}^2\to\infty$).

Another fundamental case arises when the asymmetry distribution follows a uniformly random pattern in the radial as well as the circumferential directions. In that situation, the resonance probability becomes the ratio of the area of the resonance segment to the area of the circle:

$$P=(1/\pi)\sec^{-1}|\mathcal{L}|-(1/\pi)\sqrt{(1/\mathcal{L})^2-(1/\mathcal{L})^4} \tag{2}$$

Since the first term on the right side of Eq. (2) is identical to Eq. (1), and the second term always conveys a negative quantity, the probability of resonance for the case of uniform radial distribution duly changes to somewhat less than that of the constant magnitude case.

A more appropriate apportionment of the configurational irregularities involves a radial distribution in which the chance of realizing asymmetry of extreme measure is small. Using the Gaussian or bivariate normal function, where the integral of the function $\Psi(u,v)$ represents the likelihood of experiencing an event in the interval u_iv_i, the probability equations associated with the sector of resonance p_s and the circle of possibility p_c take the forms

$$p_s=(1/\pi)\int_{u=a}^{b}\int_{v=0}^{(b^2-u^2)^{1/2}}\exp(-\tfrac{1}{2}u^2-\tfrac{1}{2}v^2)\,dv\,du \tag{3}$$

and

$$p_c=(2/\pi)\int_{u=0}^{b}\int_{v=0}^{(b^2-u^2)^{1/2}}\exp(-\tfrac{1}{2}u^2-\tfrac{1}{2}v^2)\,dv\,du \tag{4}$$

where $0<a<b$. The reference system for Eqs. (3) and (4) has been realigned such that the body-fixed u, v axes coincide with the orthogonal vectors δ_{crt}, F, respectively.

By transferring from rectangular to polar coordinates, Eq. (4) can be analytically integrated: $p_c=1-\exp(-b^2/2)$. Then, with the resonance probability here defined as the ratio p_s/p_c, the likelihood of encountering persistent resonance for designs with normally distributed asymmetries reduces to

$$P=\frac{\sqrt{2}}{\pi(1-e^{-\frac{1}{2}b^2})}\left(\int_{u=a}^{b}e^{-\frac{1}{2}u^2}\sum_{k=0}^{\infty}\frac{(-1)^k[\frac{1}{2}(b^2-u^2)]^{\frac{1}{2}(2k+1)}}{k!(2k+1)}\,du\right) \tag{5}$$

or, with a change of the variable $\xi=\sqrt{(b^2-u^2)/2}$,

$$P=\frac{\sqrt{2/\pi}}{1-e^{\frac{1}{2}b^2}}\int_{\xi=0}^{\sqrt{\frac{1}{2}(b^2-a^2)}}\frac{\xi\,\mathrm{erf}(\xi)\exp(\xi^2-\frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}}\,d\xi \tag{6}$$

which is suitable for evaluation by numerical procedure. The upper integration limit in Eq. (5) represents a dimensionless form of the maximum possible rolling moment, and, for present purposes, is assigned the "three-sigma" value, i.e., three standard deviations. Hence, the integral extends from $a=3/|\mathcal{L}|$ to $b=3$. Accordingly, the upper limit in Eq. (6) becomes $3\sqrt{(\mathcal{L}^2-1)/(2\mathcal{L}^2)}$, where, as previously indicated, $\mathcal{L}^2\geq 1$.

The latter formulas, Eqs. (5) and (6), are termed *conditional* equations because, in defining P as the ratio p_s/p_c, the distribution scales are concurrently readjusted, with the result that the maximum possible torque $\mathcal{L}$ justly reflects what its title implies, a parameter with a confining boundary. Mathematically, or otherwise, no case exceeding the finite number $|\mathcal{L}|$ is recognized in the population accounting process. Selected points obtained from the final equation are P=0.00115, 0.0197, 0.0636, 0.156, 0.224, 0.272, 0.333, 0.381, 0.440, 0.470, 0.488, and 0.499 with $|\mathcal{L}|$=1.1, 1.5, 2, 3, 4, 5, 7, 10, 20, 40, 100, and 1000, respectively. Using Eq. (6), the resonance probability is found to be less than that given by Eqs. (1) or (2), as expected. All three expressions have identical asymptotes ($P=\frac{1}{2}$).

To include in the probability estimate the combined effect of roll torques arising from both in-plane and out-of-plane asymmetry sources, an additional sector of resonance is introduced. The new sector, oriented perpendicular to the one shown in Fig. 1 and dimensioned such that the ratio of the larger of the critical torques to the smaller torque has the value $k=a/c>1$, results in a partitioned probability equation;

if $1 \le \mathcal{L}^2 \le 1+(1/k)^2$,

$$P = \frac{\sqrt{2/\pi}}{1-e^{\frac{1}{2}b^2}} \int_{\xi=0}^{\sqrt{\frac{1}{2}(b^2-a^2)}} \frac{\xi \operatorname{erf}(\xi) \exp(\xi^2 - \frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}} \mathrm{d}\xi + \frac{\sqrt{2/\pi}}{1-e^{\frac{1}{2}b^2}} \int_{\xi=0}^{\sqrt{\frac{1}{2}(b^2-c^2)}} \frac{\xi \operatorname{erf}(\xi) \exp(\xi^2 - \frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}} \mathrm{d}\xi \tag{7a}$$

and, if $\mathcal{L}^2 > 1+(1/k)^2$,

$$P = \frac{\sqrt{2/\pi}}{1-e^{-\frac{1}{2}b^2}} \int_{\xi=0}^{\sqrt{\frac{1}{2}(b^2-a^2)}} \frac{\xi \operatorname{erf}(\xi) \exp(\xi^2 - \frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}} \mathrm{d}\xi + \frac{\sqrt{1/(2\pi)}}{1-e^{-\frac{1}{2}b^2}} \left(\int_{\xi=0}^{\sqrt{\frac{1}{2}(b^2-c^2)}} \frac{\xi \operatorname{erf}(\xi) \exp(\xi^2 - \frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}} \mathrm{d}\xi + \int_{\xi=\sqrt{\frac{1}{2}(b^2-a^2)}}^{b/\sqrt{2}} \frac{\xi [\operatorname{erf}(\xi) - \operatorname{erf}(c/\sqrt{2})] \exp(\xi^2 - \frac{1}{2}b^2)}{(b^2-2\xi^2)^{\frac{1}{2}}} \mathrm{d}\xi \right) \tag{7b}$$

where $\mathcal{L}$ now denotes the asymmetry rolling moment divided by the larger of the critical torques. Equations (7a) and (7b) were evaluated with $a=3/|\mathcal{L}|$, $b=3$ and $c=3/(k|\mathcal{L}|)$. Figure 2 describes the conditional function, which has $P=\frac{3}{4}$ as its asymptote. It is found that if the asymmetry moment exceeds the critical torque by even a small amount, the chance of experiencing sustained resonance is not remote. This quantifies the common observation that those phenomena are not unusual events in missile development.

It is noted that such extreme asymmetry may exist that the angular acceleration drives the roll rate above the critical frequency, thereby avoiding resonance. But these random, supercritical asymmetries may mean either contending with sufficiently high roll rates to introduce significant gyroscopic forces or undergoing large impact dispersion, which arises when a vehicle rolls through zero rate. In any event, these probability equations report the composite anomaly of persistent resonance and pronounced asymmetry effects.

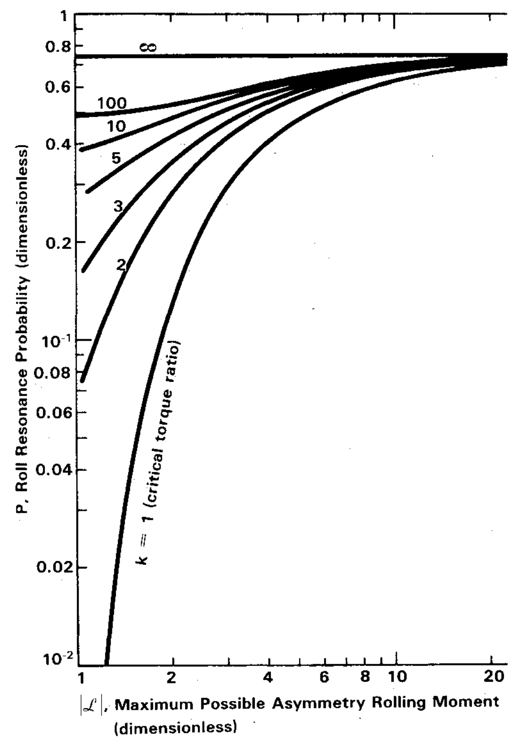

Fig. 2 Probability of a transient roll resonance event becoming one of continuous resonance during ballistic flight ($P=0$ if $|\mathcal{L}|<1$).

Summary and Conclusions

To help assess the reliability of ballistic bodies in free flight, an integral equation was developed that expresses a vehicle's susceptibility to continuous roll resonance. It assumes that configurational irregularities evolve randomly, with a Gaussian or normal distribution. Although the equation has no obvious analytical solution, two governing parameters emerge: a dimensionless form of the configuration's excess asymmetry torque over that needed to maintain resonance, and the ratio of the critical rolling moments due to in-plane and out-of-plane asymmetries. Numerical solutions indicate that with a moment ratio of two, for example, an average of one out of approximately six flights will be expected to undergo persistent resonance if the maximum possible asymmetry torque exceeds the critical amount by 50%. As the configurational abnormalities grow larger, the resonance probability markedly increases, with the result that three out of four flights ultimately become involved in either resonance or the adverse effects of random supercritical asymmetries.

Acknowledgment

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